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J. Phys. A: Math. Gen. 25 (1992) L1245–L1250. Printed in the UK

LETTER TO THE EDITOR

The Poisson bracket for q-deformed systems

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Received 4 June 1992

Abstract. It is shown that the quantum system of q-deformed oscillators with the $U_q(n)$ group symmetry can be obtained by the canonical quantization of a classical system with a modified Poisson bracket. The modification of the Poisson bracket is connected with a non-canonical transformation of phase space variables. In this approach, the deformation parameter turns out to be a function of the Planck constant and some dimensional parameters characterizing the classical system.

The idea to associate the q-deformed oscillator [1–3] with the non-standard Heisenberg commutation relation [4] attracts much attention [5–7]. In this approach, the parameter of the q-deformation becomes a function of physical constants characterizing a considered system [6]. In the present letter we consider a quantum system of oscillators with the $U_q(n)$ group symmetry. The q-deformed universal enveloping algebra $U_q(n)$ [8,9] is realized by operators being functions of the standard canonically conjugated variables satisfying the Heisenberg commutation relations. The parameter of the q-deformation turns out to be a function of the Planck constant, the oscillator frequency and a fundamental length that determines the volume of the configuration space of the system [10]. Having the quantum theory with the standard canonical variables, we take the classical limit in accordance with the rule of canonical quantization, $-i\hbar^{-1}[,] \to \{,\}$ as $\hbar \to 0$, and obtain the Poisson bracket $\{,\}$ of the 'q-deformed' holomorphic variables corresponding to the q-deformed creation and destruction operators. The latter allows us to establish the $U_q(n)$ Poisson bracket structure in the classical theory. We observe that the q-deformed canonical variables are connected with usual phase space variables through a special non-canonical transformation. Based on this, we state that the q-deformation can be associated with non-canonical transformations.

The q-bosonic oscillator realization of $U_q(n)$ is given by the following generators [9]:

$$e_{ij} = a_i^+ a_j \qquad i \neq j = 1,2,\ldots,n \tag{1}$$

$$d_i = N_i - N_{i+1} \qquad i = 1,2,\ldots,n-1 \tag{2}$$

$$d = N = \sum_{i=1}^{n} N_i \tag{3}$$

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0305-4470/92/221245+06$07.50 © 1992 IOP Publishing Ltd

where

$$a_i a_j^+ - q^{\delta_{ij}} a_j^+ a_i = \delta_{ij} q^{-N_i} \qquad a_i^+ a_i = \frac{q^{N_i} - q^{-N_i}}{q - q^{-1}} \tag{4}$$

and $[a_i, a_j] = [a_i^+, a_j^+] = 0$ and q is assumed to be a real number. It is easy to be convinced that

$$[N_i, a_j] = -a_i \delta_{ij} \qquad [N_i, a_j^+] = a_i^+ \delta_{ij} \tag{5}$$

i.e. N_i is the number operator.

Consider a simple transformation of the algebra (4), namely

$$b_i = q^{N_i/2} a_i \qquad b_i^+ = a_i^+ q^{N_i/2}. \tag{6}$$

The operators (6) satisfy the commutation relations

$$b_i b_j^+ - q^{2\delta_{ij}} b_j^+ b_i = \delta_{ij} \qquad [b_i, b_j] = [b_i^+, b_j^+] = 0 \tag{7}$$

and

$$b_i^+ b_i = \frac{1 - q^{2N_i}}{1 - q^2}. \tag{8}$$

The generators of $\mathrm{U}_q(n)$ can be written via the operators (6),

$$e_{ij} = \sqrt{q}\, b_i^+ b_j q^{N_{ij}/2} \qquad N_{ij} = N_i + N_j$$

and d_i, d keep their form (2), (3) with N_i being defined by (8). Macfarlane [3] proposed a representation of the algebra (7) (for one degree of freedom) in the space of functions of one real variable. Below we will use this represenatation.

Let us suppose now that the operators b_i and b_i^+ are the destruction and creation operators, respectively, for a system of n quantum q-deformed oscillators. We restore the Planck constant in the right-hand side of (7), $\delta_{ij} \to \hbar\delta_{ij}$. The system of n usual non-interacting oscillators with equal frequencies has the $U(n)$ symmetry generated by the operators (1)–(3) with $q = 1$. This means that the Hamiltonian $H = \hbar\omega \sum b_i^+ b_i - \hbar\omega n/2$, where ω is the frequency, commutes with all the generators of $U(n)$. As has been noted in [11], there is no system of n *non-interacting* q-deformed oscillators with the $\mathrm{U}_q(n)$ symmetry. The reason is simple: the free Hamiltonian $H = \hbar\omega \sum b_i^+ b_i - \hbar\omega n/2$ does not commute with the generators e_{ij} of $\mathrm{U}_q(n)$ and, hence, its spectrum has the broken $\mathrm{U}_q(n)$ symmetry. Therefore, if we wish to keep the q-analogy of the $U(n)$ symmetry after the q-deformation of all oscillators, we have also to modify the Hamiltonian.

The operator $H_q = \hbar\omega(N - n/2)$ commutes with all the generators of $\mathrm{U}_q(n)$. We can take it as the $\mathrm{U}_q(n)$-invariant Hamiltonian because it coincides with the free Hamiltonian in the limit $q \to 1$. In this case, just a self-interaction of each oscillator appears due to the non-linearity of the relation (8). However, *a priori* there is no restriction for choosing the Hamiltonian, except the requirement for its behaviour in the limit $q \to 1$. So, a non-linear function $H_q = H_q(N)$ can also serve as the $\mathrm{U}_q(n)$-invariant Hamiltonian if $H_q \to H$ as $q \to 1$. The latter leads to an interaction of oscillators. Note that the operator $H_q = \hbar\omega(N - n/2)$ has the same spectrum as the Hamiltonian of n usual oscillators. We shall not choose a concrete form of the $U_q(n)$-invariant Hamiltonian.

Physicists have made a 'deformation' of fundamental physical laws several times. For instance, relativistic as well as quantum mechanics can be considered as deformations of classical mechanics with the deformation parameters v/c, where c is the light velocity, and $S/\hbar$, where S is an action of a system, respectively. It is believed that the deformation parameter q should be associated with a new fundamental dimensional constant like c or $\hbar$. Below we show how to realize this program for a system of bosonic q-oscillators.

Consider the following representation of the operators (7) [3, 10]:

$$\begin{aligned} b_j &= \alpha\left(e^{-2i\hat{x}_j/l_q} - e^{-i\hat{x}_j/l_q}e^{-\hat{p}_j/(\omega l_q)}\right) \\ b_j^+ &= \bar{\alpha}\left(e^{2i\hat{x}_j/l_q} - e^{-\hat{p}_j/(\omega l_q)}e^{i\hat{x}/l_q}\right) \end{aligned} \tag{9}$$

where

$$[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk} \tag{10}$$

and, therefore, [10] $\alpha\bar{\alpha} = \hbar(1 - q^2)^{-1}$,

$$q = \exp\left(-\frac{\hbar}{\omega l_q^2}\right) \tag{11}$$

with l_q being a dimensional parameter (a fundamental length). The opearators (9) are conjugated to each other in the Hilbert space of functions of n real variables x_j if $\hat{x}_j = x_j$ and $\hat{p}_k = -i\hbar\partial/\partial x_k$ [3].

The equalities (9) define the q-deformed canonical variables via the usual ones obeying the standard Heisenberg commutation relations (10). This allows us to construct the corresponding q-deformed classical theory with the Poisson bracket on the phase space of commutative variables (x_j, p_j). We can achieve this by taking the formal classical limit $\hbar \to 0$ for the q-deformed canonical variables (9) so that the opearators $\hat{x}_j$, $\hat{p}_j$ become the usual commutative canonical variables x_j, p_j. Note that $q \to 1$ as $\hbar \to 0$ and, hence, the q-deformed canonical variables (9) are also commutative in the classical limit. So, we arrive at the classical theory with the non-ordinary canonical variables [10]

$$\begin{aligned} b_j &= l_q\sqrt{\frac{\omega}{2}}\left(e^{-2ix_j/l_q} - e^{-ix_j/l_q}e^{-p/(\omega l_q)}\right) \\ b_j^* &= l_q\sqrt{\frac{\omega}{2}}\left(e^{2ix_j/l_q} - e^{ix_j/l_q}e^{-p/(\omega l_q)}\right) \end{aligned} \tag{12}$$

where we take into account $\alpha\bar{\alpha} \to \omega l_q^2/2$ as $\hbar \to 0$.

One should stress that the formal limit $\hbar \to 0$, inspired by the rule of the canonical quantization, $-i\hbar^{-1}[\hat{x}_j, \hat{p}_k] \to \{x_j, p_k\}$ as $\hbar \to 0$ ($\{,\}$ is the Poisson bracket), cannot be considered as a well founded way of derivation of the classical theory because the operators (9) have a rather complicated dependence on $\hbar$. It is necessary to consider more carefully a semiclassical approximation of the q-deformed theory in the representation (9). This has been done in [10] by investigating the semiclassical limit of the path integral for the transition amplitude $\langle x|\exp(-it\hat{H}_q/\hbar)|x'\rangle$ where $\hat{H}_q = H_q(b^+, b)$. In this approximation, the quantity $H_q = H_q(b^*, b)$, where b^*, b are defined by (12), plays the role of the classical Hamiltonian [10]. Because of this property, we can identify the variables (12) with the classical limit of the operators (9).

There is another feature of the 'q-deformed' classical theory, which cannot be directly observed in the formal limit $\hbar \to 0$ but it naturally appears in the path-integral approach. It is the compactification of the configuration space [10]. This means that x_j take their values on the interval $\Omega_l = [-\pi l_q/2,\ \pi l_q/2]$ with the identified boundary points $\pm\pi l_q/2$, i.e. $x_j \in S^1$ (a circle). When the 'volume' of the configuration space tends to infinity, $l_q \to \infty$, the variables (9) and (12) as well turn into the usual canonical ones.

Another way to establish the correspondence between (12) and (9) is to consider the canonical quantization of the variables (12), i.e. to change the canonical variables $x_j,\ p_j$ by the operators with the commutation relation (10). The operator ordering problem can be solved with the help of the requirement $(\tilde{b}_j^+)^+ = \tilde{b}_j$ where $\tilde{b}_j$ is the operator obtained by the substitution $x_j, p_j \to \hat{x}_j, \hat{p}_j$ into the classical quantity b_j (12). It is easy to see that

$$\tilde{b}_i\tilde{b}_j^+ - q^{2\delta_{ij}}\tilde{b}_j^+\tilde{b}_i = \frac{\omega l_q^2}{2}\left(1-q^2\right)\delta_{ij} \qquad q = \exp\left(-\frac{\hbar}{\omega l_q^2}\right). \tag{13}$$

Therefore, putting

$$\tilde{b}_i = \beta b_i \qquad \tilde{b}_i^+ = \bar{\beta} b_i^+ \qquad \beta\bar{\beta} = \frac{\omega l_q^2}{2\hbar}(1-q^2) \tag{14}$$

we obtain the quantum theory with the destruction and creation operators obeying the algebra (7).

The equalities (12) determine a *non-canonical* transformation $x_j, p_j \to X_j, P_j$,

$$X_j = X_j(x,p) = \frac{i}{\sqrt{2\omega}}\left(b_j - b_j^*\right) \qquad P_j = P_j(x,p) = \sqrt{\frac{\omega}{2}}\left(b_j + b_j^*\right) \tag{15}$$

$$\{X_i,\ P_j\} = \delta_{ij}\exp\left(-\frac{H_i}{E_l}\right) \qquad E_l = \frac{\omega^2 l_q^2}{2} \tag{16}$$

where the quantity H_i can be treated as the classical limit of the operator $\hbar\omega N_i$ and defined by the classical analogy of the equality (8)†

$$\omega b_i^* b_i = \tfrac{1}{2}\left(P_i^2 + \omega^2 X_i^2\right) = E_l\left(1 - e^{-H_i/E_l}\right). \tag{17}$$

The functions H_i have the remarkable properties

$$\{X_i,\ H_j\} = \delta_{ij}P_i \qquad \{P_i,\ H_j\} = -\delta_{ij}\omega^2 X_i \tag{18}$$

† Note that for a usual harmonic oscillator, the classical limit means that the eigenvalue of the the number operator tends to infinity, $N_i \to \infty$, but the operator $\hbar\omega N_i$ turns into the classical oscillator Hamiltonian as $\hbar \to 0,\ N_i \to \infty$.

which allows us to integrate the classical equations of motion [12] because the classical Hamiltonian is a function of $H = \sum_{i=1}^{n} H_i$ as has been noted previously.

Thus, we are convinced that the q-deformed Heisenberg commutation relations can be obtained by the canonical quantization, $[,] = \mathrm{i}\hbar\{,\}$, of phase space variables connected with the canonical ones by a special non-canonical transformation.

The relations (16) and (17) define the symplectic geometry on the phase space $(X_i,\ P_i)$ with the symplectic form (16). Having the Hamiltonian formalism, one can also derive the corresponding Lagrangian formalism with the help of the Legendre transformation.

Finishing our discussion, we give the explicit Poisson bracket structure of the generators of $\mathrm{U}_q(n)$. We define the generators as the following functions on the phase space $(X_i,\ P_i)$ (or on the phase space of the canonical variables $(x_i,\ p_i)$)

$$e_{mj} = b_m^* b_j \qquad m \neq j = 1, 2, \ldots, n \tag{19}$$

$$d_j = \frac{1}{\omega}\left(H_j - H_{j+1}\right) \qquad j = 1, 2, \ldots, n-1 \tag{20}$$

$$d = \frac{1}{\omega} H = \frac{1}{\omega}\sum_{j=1}^{n} H_j\,. \tag{21}$$

The commutation relations with respect to the Poisson bracket (16) read

$$\{e_{jk},\ e_{j'k'}\} = -\mathrm{i}\left(\delta_{j'k} e_{jk'} \mathrm{e}^{-H_k/E_l} - \delta_{jk'} e_{j'k} \mathrm{e}^{-H_j/E_l}\right) \tag{22}$$

$$\{e_{jk},\ d_m\} = -\mathrm{i}\left(\delta_{km} - \delta_{jm} - \delta_{km+1} + \delta_{jm+1}\right) e_{jk} \tag{23}$$

$$\{e_{jk},\ d\} = 0 \qquad \{d_j,\ d\} = 0\,. \tag{24}$$

In the case of $j = k'$ or $j' = k$ in the right-hand side of the equality (22), the quantity e_{jj} must be treated as $b_j^* b_j$ defined by (17). The Hamiltonian, being a function of H, obviously commutes with all the generators (19)–(21). After the canonical quantization of (19)–(21), we get the q-bosonic oscillator realization of $\mathrm{U}_q(n)$ (equations (1)–(3)).

The other examples of classical systems with the $\mathrm{SU}_q(2)$ Poisson bracket structure are considered in [13] (an oscillator with the broken $\mathrm{SU}_q(2)$ symmetry) and [14].

One can introduce the vector fields V_σ associated with the generators (19) such that $V_\sigma F = \{\sigma,\ F\}$ where σ runs over the set $(e_{ij},\ d_j,\ d)$ and F is an arbitrary function on the phase space. Due to the property $[V_\sigma,\ V_{\sigma'}] = V_{\{\sigma,\sigma'\}}$ guaranteed by the Jacobi identity for the Poisson bracket (16), the vector fields V_σ realize a representation of $\mathrm{U}_q(m)$ in a space of functions on the phase space spanned commutative coordinates.

This work was supported by the Tomalla Foundation.

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